Exercises 26.3

Download CLICAL from

`www.teli.stadia.fi/~lounesto/CLICAL.htm`

Problem 1: Determine the distance d of the point P from the line AB, when $A = (2, 3, 5)$, $B = (1, 6, 3)$, $P = (5, -1, 3)$.
Solution: $d = |((P - A) \wedge (B - A))/(B - A)|$.
Explanation: The area of a parallelogram, which is twice as big as the triangle ABP, is divided by the line segment AB.
Comment: This formula is applicable in any dimension. ∎

Problem 2: Find the distance d between two lines, say AB and CD, when $A = (2, 5, 3)$, $B = (4, 1, 3)$, $C = (5, 2, 1)$, $D = (7, 3, 4)$.
Solution: $d = |((A - C) \wedge E)/E|$, where $E = (A - B) \wedge (C - D)$.
Explanation: Determine the length of the orthogonal rejection of $A - C$ outside of the plane $E = (A - B) \wedge (C - D)$.
Comment: This formula is independent of the surrounding dimension. ∎

Problem 3: Find the line of intersection, say $\ell(t)$, of the two planes, $\mathbf{v}_1 \cdot \mathbf{r} + d_1 = 0$ and $\mathbf{v}_2 \cdot \mathbf{r} + d_2 = 0$, when $\mathbf{v}_1 = 3\mathbf{e}_1 + 4\mathbf{e}_2 + 2\mathbf{e}_3$, $d_1 = 2$ and $\mathbf{v}_2 = 5\mathbf{e}_1 + 3\mathbf{e}_2$, $d_2 = 3$.
Solution: $\ell(t) = t(\mathbf{v}_1 \wedge \mathbf{v}_2)/\mathbf{e}_{123} + (d_2\mathbf{v}_1 - d_1\mathbf{v}_2)/(\mathbf{v}_1 \wedge \mathbf{v}_2)$.
Comment: A solution, using the cross product, can be written as

$$\ell(t) = t(\mathbf{v}_1 \times \mathbf{v}_2) + (d_2\mathbf{v}_1 - d_1\mathbf{v}_2) \times (\mathbf{v}_1 \times \mathbf{v}_2)/|\mathbf{v}_1 \times \mathbf{v}_2|^2.$$

For an arbitrary vector $\mathbf{c}$, $\mathbf{c}/(\mathbf{v}_1 \wedge \mathbf{v}_2)$ is a sum of a 3-vector and a 1-vector, which equals $\mathbf{c} \times (\mathbf{v}_1 \times \mathbf{v}_2)/|\mathbf{v}_1 \times \mathbf{v}_2|^2$.
Explanation: The quotient A/B is the orthogonal complement of A within B, with magnitude $|A|/|B|$. Thus, $(\mathbf{v}_1 \wedge \mathbf{v}_2)/\mathbf{e}_{123}$ is a vector orthogonal to the plane $\mathbf{v}_1 \wedge \mathbf{v}_2$ within the $\mathbf{e}_{123}$-space $\mathbb{R}^3$, and $(d_2\mathbf{v}_1 - d_1\mathbf{v}_2)/(\mathbf{v}_1 \wedge \mathbf{v}_2)$ is the orthogonal complement of the vector $d_2\mathbf{v}_1 - d_1\mathbf{v}_2$ within the plane $\mathbf{v}_1 \wedge \mathbf{v}_2$.
Comment: The intersection of two 3-planes in $\mathbb{R}^4$ is a 2-plane, determined by its bivector $(\mathbf{v}_1 \wedge \mathbf{v}_2)/\mathbf{e}_{1234}$ and a position vector $(d_2\mathbf{v}_1 - d_1\mathbf{v}_2)/(\mathbf{v}_1 \wedge \mathbf{v}_2)$.
∎

Problem 4: Find out, if a line segment intersects a plane in $3D$-space; if so, at what point does the intersection occur; what is the distance between each endpoint and the intersection. Assume that the plane contains the

points $S = (7, -7, 6)$, $T = (1, 3, 2)$, $O = (0, 0, 0)$ and assume that the line segment has endpoints $A = (3, -4, 7)$ and $B = (2, 4, 1)$.
Solution: While in CLICAL type

```
> dim 3
> S = 7e1-7e2+6e3
> T = e1+3e2+2e3
> P = S^T
P = 28e12+8e13-32e23    [this represents the plane]

> A = 3e1-4e2+7e3
> B = 2e1+4e2+e3
> (A^P)/((A-B)^P)
ans = 0.660         [intersection occurs, since this is between 0..1]

> C = A+(B-A)*ans
C = 2.340e1+1.280e2+3.040e3  [the point of intersection]

> abs(A-C)
ans = 6.633         [distance of an endpoint from the intersection]
> abs(B-C)
ans = 3.417         [distance of an endpoint from the intersection]
```

Explanation: $A \wedge P = A \wedge S \wedge T$ is the oriented volume of the parallelepiped with A, S, T as edges. ∎

Problem 5: A person looks at a tetrahedron with corners A, B, C, D from the position P. Is the face ABC with vertices A, B, C visible to the person at P? $A = (1, 2, 3)$, $B = (3, 7, 1)$, $C = (2, 0, 0)$, $D = (2, 3, 6)$, $P = (6, 6, 6)$,
Solution: No, for opaque faces or interior. In CLICAL, treat A, B, C, D and P as vectors:

```
> dim 3
> A = e1+2e2+3e3
> B = 3e1+7e2+e3
> C = 2e1
> D = 2e1+3e2+6e3
> P = 6e1+6e2+6e3
```

```
> ((D-A)^(C-A)^(B-A))/((C-A)^(B-A))
ans = 1.742e1-0.367e2+0.825e3

> ((P-A)^(C-A)^(B-A))/((C-A)^(B-A))
ans = 4.397e1-0.925e2+2.083e3
```

Since the two vectors, the first `ans` and the second `ans`, point to the same direction, P and D are on the same side of the plane ABC. Thus, the person at P cannot see the face ABC.
Comment: The two answers compute the orthogonal rejections (outside of ABC) of the vectors $D-A$ and $P-A$. The plane ABC is represented by the bivector $(C-A)\wedge(B-A) = 9\mathbf{e}_{12} + 4\mathbf{e}_{13} + 19\mathbf{e}_{23}$. ∎

Problem 6: Determine the angle ABC for $A = (5,9)$, $B = (2,3)$, $C = (8,3)$.
Solution: For complex numbers A, B, C: angle $= |\mathrm{Im}(\log((A-B)/(C-B)))|$. In CLICAL, you can treat A,B,C also as vectors:

```
> A = 5e1+9e2
> B = 2e1+3e2
> C = 8e1+3e2
> log((A-B)/(C-B))
ans = 0.122-1.107i
```

Thus, the angle is 1.107.
Comment: This method is generalizable to higher dimensions in the form angle $= |\langle\log((A-B)/(C-B))\rangle_2|$, where $\langle W\rangle_2$ gives the bivector part of W (computable in CLICAL as $\mathrm{Pu}(2, W)$). ∎

Problem 7: Find a rotation sending a unit vector $\mathbf{x}$ to the unit vector $\mathbf{y}$.
Solution: $\mathbf{y} = u\mathbf{x}/u$, where $u = \sqrt{\mathbf{y}/\mathbf{x}}$.
Explanation: In $2D$, $\mathbf{xr}/\mathbf{x}$ is the vector $\mathbf{r}$ reflected across the line $\mathbf{x}$; and $(\mathbf{xy})\mathbf{r}/(\mathbf{xy})$ is $\mathbf{r}$ reflected first across $\mathbf{y}$ and then across $\mathbf{x}$; this means rotation by twice the angle between $\mathbf{x}$ and $\mathbf{y}$; thus the desired rotation is completed by $u = \sqrt{\mathbf{y}/\mathbf{x}}$. Recall that $1/\mathbf{x} = \mathbf{x}$ and $\mathbf{xrx} = \mathbf{x}(\mathbf{r}\cdot\mathbf{x} + \mathbf{r}\wedge\mathbf{x}) = \mathbf{x}(\mathbf{x}\cdot\mathbf{r} - \mathbf{x}\wedge\mathbf{r}) = \mathbf{x}(2\mathbf{x}\cdot\mathbf{r} - \mathbf{xr}) = 2(\mathbf{x}\cdot\mathbf{r})\mathbf{x} - (\mathbf{xx})\mathbf{r}$, which means reflection of $\mathbf{r}$ across $\mathbf{x}$.
Comment: The formula is valid in any dimension. ∎

Problem 8: What is the distance of two 2-planes in $5D$, with no common points? Say, for instance of the planes

A : spanned by $a_1 = \mathbf{e}_2 + 5\mathbf{e}_5$ and $a_2 = \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_4$
B : spanned by $b_1 = 2\mathbf{e}_1 - 2\mathbf{e}_2 - 4\mathbf{e}_3 - 3\mathbf{e}_4$ and $b_2 = 2\mathbf{e}_2 + 3\mathbf{e}_3 + 4\mathbf{e}_4 + \mathbf{e}_5$

where A passes through the origin 0 and B through $c = \mathbf{e}_1 + 3\mathbf{e}_3 - \mathbf{e}_4 + 7\mathbf{e}_5$.
Solution: While in CLICAL, type

```
> dim 4

input the data, and type

> A = a1^a2
A = -e12-5e15+e24-5e25-5e45
> B = b1^b2
B  = 4e12+6e13+8e14+2e15+2e23-2e24-2e25-7e34-4e35-3e45

compute the component of c perpendicular to both A and B

> u = (c^A^B)/(A^B)
u = 1.793e1-0.309e2+2.165e3-1.484e4

where u is the projection of c in that perpendicular direction.
The required distance is the length of u,

> d = abs(u)
ans = 3.194413
```

Comment: The above construction works in any dimension n for computing the distance of two planes. If you want to benefit $n = 5$, you could also compute $v = (A \wedge B)\mathbf{e}_{12345} = 29\mathbf{e}_1 - 5\mathbf{e}_2 + 35\mathbf{e}_3 - 24\mathbf{e}_4 + \mathbf{e}_5$, project c on v to get $u = (c \cdot v)/v$; and $d = |u| = 3.194$. ∎

Problem 9: Determine the principal angles between two 2-planes in $\mathbb{R}^4$, the planes being the xy-plane and the plane spanned by $(1, 0, 1, 0)$ and $(0, 1, 0, 7)$.
Comment: The principal angles between these two planes are $45°$ and $81.9° = \arctan(7)$. This means that two lines in the two planes are separated by at least $45°$, and at this minimum, the orthogonal complements of the two lines are separated by $81.9°$.

Denote the xy-plane $\mathbf{e}_{12}$ by A and the other plane by

$$B = (\mathbf{e}_1 + \mathbf{e}_3) \wedge (\mathbf{e}_2 + 7\mathbf{e}_4) = \mathbf{e}_{12} + 7\mathbf{e}_{14} - \mathbf{e}_{23} + 7\mathbf{e}_{34}.$$

Then $B/A = 1 - \mathbf{e}_{13} - 7\mathbf{e}_{24} - 7\mathbf{e}_{1234}$, computed by observing that $AA = -1$. Then $\log(B/A) = \log(10) - (\pi/4)\mathbf{e}_{13} - \arctan(7)\mathbf{e}_{24}$, as can be verified by exponentiation (and observing that $\mathbf{e}_{13}\mathbf{e}_{24} = \mathbf{e}_{24}\mathbf{e}_{13}$). Thus, the two principal angles, $\pi/4$ and $\arctan(7)$, occur in the pure bivector part of $\log(B/A)$,

$$\mathbf{F} = \mathrm{Pu}(2, \log(B/A)) = -(\pi/4)\mathbf{e}_{13} - \arctan(7)\mathbf{e}_{24}.$$

The bivector $\mathbf{F}$ decomposes into a sum of two simple bivectors,

$$\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2, \quad \mathbf{F}_1 = -\arctan(7)\mathbf{e}_{24}, \mathbf{F}_2 = -(\pi/4)\mathbf{e}_{13}.$$

The principal angles f_1 and f_2 are the magnitudes of $\mathbf{F}_1$ and $\mathbf{F}_2$,

$$f_1 = |\mathbf{F}_1|, \quad f_2 = |\mathbf{F}_2|.$$

The problem is to find a formula to for the decomposition $\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2$.
Solution: $\mathbf{F}_1 = \mathbf{F}\langle\sqrt{-\mathbf{F}^2}\rangle_0/\sqrt{-\mathbf{F}^2}$,

$$\mathbf{F}_2 = \mathbf{F}\langle\sqrt{-\mathbf{F}^2}\rangle_4/\sqrt{-\mathbf{F}^2},$$

computable with CLICAL as follows:

```
> dim 4
> A = e12
> B = (e1+e3)^(e2+7e4)
B = e12+7e14-e23+7e34

> B/A
ans = 1-e13+7e24-7e1234

> F = Pu(2,log(ans))
F = -0.785e13-1.429e24

> F1 = F*Pu(0,sqrt(-F**2))/sqrt(-F**2)
F1 = -1.429e24

> F2 = F*Pu(4,sqrt(-F**2))/sqrt(-F**2)
F2 = -0.785e13
```

Thus, the principal angles are $f_1 = 1.429$ and $f_2 = 0.785$.
Comment: The mutual attitude of two lines is determined by the angle between them. The mutual attitude of two k-dimensional subspaces, in n dimensions, is given by k angles, one being the smallest angle between any directions in the two subspaces (another being the smallest angle between the remaining $(k-1)$-dimensional subspaces, where the first directions have been rejected). ∎

Problem 10. Find the distance of the point $P = (2, 3, 1)$ from the line AB, where $A = (1, 2, 0)$, $B = (3, 0, -2)$.

```
> dim 3
> P = 2e1+3e2+e3
> A = e1+2e2
> B = 3e1-2e3
> abs(((P-B)^(A-B))/(A-B))
ans = 1.633   [= sqrt(8/3)]
```

∎

Problem 11. Compute $i/(j + \exp(k\pi/6))$ in quaternions. Hint: Go to the Clifford algebra $\mathcal{C}\ell_{0,3}$ and use the correspondences $i = \mathbf{e}_1$, $j = e_2$, $k = e_3$.

```
> dim 0,3
> q(u) = Re((1-e123)u)+Pu(1,(1-e123)u)
> q(e1/(e2+exp(pi/6 e3)))
ans = 0.433e1+0.250e2-0.5e3   [= sqrt(3)/4 i + 1/4 j - 1/2 k]
```

∎

Problem 12. Find matrices of the two isoclinic rotations (= turns each plane the same angle) $U_L(a) = b$, $U_R(a) = b$ sending $a = (16 + 12i + 5j + 4k)/21$ to $b = (18+10i+4j+k)/21$. Compute $ba^{-1} = (1/441)(432-67\mathbf{e}_1 + 2\mathbf{e}_2 - 58\mathbf{e}_3)$, $a^{-1}b = (1/49)(48 - 5\mathbf{e}_1 - 6\mathbf{e}_2 - 6\mathbf{e}_3)$ and the components of $(ba^{-1})q$, $q(a^{-1}b)$ for $q = 1, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ to find the entries of

$$U_L = \frac{1}{441}\begin{pmatrix} 432 & 67 & -2 & 58 \\ -67 & 432 & 58 & 2 \\ 2 & -58 & 432 & 67 \\ -58 & -2 & -67 & 432 \end{pmatrix},$$

$$U_R = \frac{1}{49}\begin{pmatrix} 48 & 5 & 6 & 6 \\ -5 & 48 & -6 & 6 \\ -6 & 6 & 48 & -5 \\ -6 & -6 & 5 & 48 \end{pmatrix}. \qquad \blacksquare$$

Problem 13. Find the matrix of the simple rotation (= turns only one plane) of $\mathbb{R}^4$ sending $a = (16/21, 12/21, 5/21, 4/21)$ to $b = (18/21, 10/21, 4/21, 1/21)$.

```
> dim 4
> a = (16e1+12e2+5e3+4e4)/21
> b = (18e1+10e2+4e3+e4)/21
> s = sqrt(b/a)
s = 0.995+0.064e12+0.030e13+0.064e14+0.002e23+0.032e24+0.013e34
  [= (873+56e12+26e13+56e14+2e23+28e24+11e34)/sqrt(769986)]
```

Then compute $S_{ij} = \mathbf{e}_i \cdot (s\mathbf{e}_j/s)$ to find the matrix of the simple rotation:

$$S = \frac{1}{42777}\begin{pmatrix} 42005 & 5252 & 2466 & 5638 \\ -5612 & 42341 & -2 & 2370 \\ -2578 & -390 & 42688 & 899 \\ -5226 & -3062 & -1235 & 42328 \end{pmatrix}. \qquad \blacksquare$$

Problem 14. Form the matrix of the rotation, which is a composition of the four hyperplane reflections along $\mathbf{e}_4 - \mathbf{e}_3,\ \mathbf{e}_3 - \mathbf{e}_2,\ \mathbf{e}_2 - \mathbf{e}_1,\ \mathbf{e}_1$.

```
> a = e1(e2-e1)(e3-e2)(e4-e3)
a = 1-e12+e13-e14-e23+e24-e34+e1234
> a e1/a
ans = e2
> a e2/a
ans = e3
> a e3/a
ans = e4
> a e4/a
ans = -e1
```

Thus, the rotation matrix is $[000-1, 1000, 0100, 0010]$. The rotation turns one plane by angle $3\pi/4$ and its orthogonal complement by angle $\pi/4$, the latter plane being $\mathbf{e}_{12}+\sqrt{2}\mathbf{e}_{13}+\mathbf{e}_{14}+\mathbf{e}_{23}+\sqrt{2}\mathbf{e}_{24}+\mathbf{e}_{34}$. The rotation sends the plane $\mathbf{e}_{12}$ to $\mathbf{e}_{23}$, $\mathbf{e}_{23}$ to $\mathbf{e}_{34}$, $\mathbf{e}_{34}$ to $\mathbf{e}_{14}$ and $\mathbf{e}_{14}$ to $\mathbf{e}_{12}$. ▮

Problem 15. A 2-plane P in $\mathbb{R}^4$ is called a T-plane of a rotation R, if $R(P) \cup P$ is a line, the line being called a T-line and the rotation being called a T-rotation. Are all non-isoclinic rotations T-rotations? Are all lines not in the invariant planes T-lines? Are all non-invariant planes T-planes? No (simple rotations of angle π are not), yes (for simple rotations of angle π and isoclinic rotations all lines are in invariant planes) and no. Take any line L outside of the two rotation planes of a non-isoclinic rotation R and consider the plane $R^{-1}(L) \wedge L$. Then $R(R^{-1}(L) \wedge L) = L \wedge R(L)$. Through a T-line there are exactly two T-planes. ▮

Problem 16. Find the intersection of the plane $A_1A_2A_3$ and the line B_1B_2 when $A_1 = (3,4,5)$, $A_2 = (7,2,5)$, $A_3 = (2,2,7)$, and $B_1 = (4,4,7)$, $B_2 = (3,6,7)$.

```
> dim 3
> A = (A1-A3)^(A2-A3)
A = -10e12+8e13-4e23
> k = (((A1-B1)^A)/A)/(((B2-B1)^A)/A)
k = -2
> P = B1+k*(B2-B1)
P = 6e1+7e3
```

or in dimension 4

```
> dim 4
> a1 = A1+e4
> a2 = A2+e4
> a3 = A3+e4
> b1 = B1+e4
> b2 = B2+e4
> shuf(u,v) = ((u/e1234)^(v/e1234))e1234
> q = shuf(a1^a2^a3,b1^b2)
q = 72e1+84e3+12e4
> p = q/(q.e4)
p = 6e1+7e3+e4
```