

Cliffordin algebrat sähkömagnetiikassa

- 1§ Lyhyt historiikki
- 2§ Diracin operaattori euklidisissa avaruuksissa
- 3§ Indeksinotaatiosta Cliffordin algebraan
- 4§ Variaatiolaskento

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- Yleensä algebroissa $\mathcal{C}\ell_{1,3}/\mathcal{C}\ell_{3,1}$ tai $\mathcal{C}\ell_{3,0}$
- Juvet ja Schidlof (1932), Mercier (1935), Riesz (1947,1958).
- Hestenes (1966), Jancewicz, Baylis (1980-luku).

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2§ Suunnattu derivaatta ja Diracin operaattori

Olkoon $f : \mathbb{R}^{p,q} \rightarrow \mathcal{C}\ell_{p,q}$, ja $\mathbf{a} \in \mathbb{R}^{p,q}$.

Suunnattu derivaatta määritellään

$$(\mathbf{a} \cdot \partial) f(\mathbf{x}) = \lim_{\tau \rightarrow 0} \frac{f(\mathbf{x} + \tau \mathbf{a}) - f(\mathbf{x})}{\tau}.$$

Diracin operaattorin voi nyt identifioida

$$\partial = \mathbf{e}^\alpha (\mathbf{e}_\alpha \cdot \partial) = \mathbf{e}^\alpha \partial_\alpha,$$

$$\text{kun } \mathbf{e}^\alpha \cdot \mathbf{e}_\beta = \delta_\alpha^\beta, \quad \text{ja } \mathbf{e}^\alpha \cdot \mathbf{e}^\beta = g^{\alpha\beta}.$$

3§ Indeksinotaatiosta...

$\boxed{\mathbb{R}^{3,1}}$

$$(F^{\alpha\beta}) = \begin{pmatrix} 0 & -B_3 & B_2 & \frac{1}{c}E_1 \\ B_3 & 0 & -B_1 & \frac{1}{c}E_2 \\ -B_2 & B_1 & 0 & \frac{1}{c}E_3 \\ -\frac{1}{c}E_1 & -\frac{1}{c}E_2 & -\frac{1}{c}E_3 & 0 \end{pmatrix}$$

eli

$$F^{\alpha\beta} = -F^{\beta\alpha},$$

$$\alpha = 1, 2, 3 : x_\alpha = x^\alpha, \quad \alpha = 4 : x_4 = -x^4$$

Tyhjössä

$$\partial_\alpha F^{\alpha\beta} = \mu_0 J^\beta,$$

$$\partial^\alpha F^{\beta\gamma} + \partial^\beta F^{\gamma\alpha} + \partial^\gamma F^{\alpha\beta} = 0.$$

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$$\partial_\alpha F^{\alpha\beta} \mathbf{e}_\beta = \partial_\alpha F^{\nu\beta} (\mathbf{e}^\alpha \cdot \mathbf{e}_\nu) \mathbf{e}_\beta = \mathbf{e}^\alpha \partial_\alpha \lrcorner \left(\frac{1}{2} F^{\nu\beta} \mathbf{e}_{\nu\beta} \right),$$

ts.

$$\partial \lrcorner \mathbf{F}.$$

Koska

$$\mathbf{J} = J^\beta \mathbf{e}_\beta$$

saadaan

$$\boxed{\partial \lrcorner \mathbf{F} = \mu_0 \mathbf{J}}$$

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$$\begin{aligned} \partial^\alpha \left[\frac{1}{6} F^{\beta\gamma} \right] \mathbf{e}_{\alpha\beta\gamma} &= 0, \\ \mathbf{e}_\alpha \partial^\alpha \wedge \left(\frac{1}{6} F^{\beta\gamma} \mathbf{e}_{\beta\gamma} \right) &= \mathbf{e}^\alpha \partial_\alpha \wedge \left(\frac{1}{6} F^{\beta\gamma} \mathbf{e}_{\beta\gamma} \right) \\ &= \partial \wedge \mathbf{F} = 0. \end{aligned}$$

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Tyhjössä voi Maxwellin yhtälöt tiivistää yhdeksi yhtälöksi:

$$\partial \lrcorner \mathbf{F} + \partial \wedge \mathbf{F} = \partial \mathbf{F} = \mu_0 \mathbf{J}$$

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Havaitsijalle \mathbf{u} kentät ovat

$$\vec{E} = \mathbf{u} \lrcorner \mathbf{F}, \quad \vec{B} = -\star(\mathbf{u} \wedge \mathbf{F}).$$

ja siten

$$\mathbf{F}|_{\mathbf{u}=c\mathbf{e}_4} = \frac{1}{c}\vec{E}\mathbf{e}_4 - \vec{B}\mathbf{e}_{123}.$$

$$\begin{aligned}\partial \wedge \mathbf{F} &= \left(\nabla - \mathbf{e}_4 \frac{1}{c} \frac{\partial}{\partial t} \right) \wedge \left(\frac{1}{c} \vec{E} \mathbf{e}_4 - \vec{B} \mathbf{e}_{123} \right) \\ &= \frac{1}{c} \mathbf{e}_{123} (\nabla \times \vec{E}) \mathbf{e}_4 - \mathbf{e}_{123} (\nabla \cdot \vec{B}) - \mathbf{e}_{1234} \frac{1}{c} \frac{\partial \vec{B}}{\partial t} = 0. \\ \partial \lrcorner \mathbf{F} &= \left(\nabla - \mathbf{e}_4 \frac{1}{c} \frac{\partial}{\partial t} \right) \lrcorner \left(\frac{1}{c} \vec{E} \mathbf{e}_4 - \vec{B} \mathbf{e}_{123} \right) \\ &= \frac{1}{c} (\nabla \cdot \vec{E}) \mathbf{e}_4 + \nabla \times \vec{B} - \frac{1}{c^2} \frac{\partial \vec{E}}{\partial t} = \mu_0 (c\rho \mathbf{e}_4 + \vec{J}).\end{aligned}$$

\Rightarrow yhtälöt tyhjössä.

Maxwellin homogenisen yhtälön ratkaisuista

Kotitehtävä:

Vakiovektorit $\mathbf{n}, \mathbf{u} \in \mathbb{R}^{1,3}$ toteuttavat ehdot $\mathbf{n}^2 = -1$,
 $\mathbf{u}^2 = 1$, ja $\mathbf{u} \cdot \mathbf{n} = 0$.

$$\partial \exp[\mathbf{e}_{1234}(\mathbf{n} \cdot \mathbf{x} + \mathbf{u} \cdot \mathbf{x})]$$

$$\partial \{ \partial \exp[\mathbf{e}_{1234}(\mathbf{n} \cdot \mathbf{x} + \mathbf{u} \cdot \mathbf{x})] \}.$$

Yleisesti:

$$\partial \wedge \mathbf{F} = 0,$$

$$\partial \lrcorner \mathbf{G} = \mathbf{J},$$

$$\mathbf{G} = \chi(\mathbf{F}).$$

Vaihtoehtoisia merkintätapoja

$$\partial \wedge \mathbf{F} = 0,$$

$$\partial \wedge \mathcal{G} = \mathcal{J},$$

$$\mathcal{G} = \star \mathbf{G}, \text{ ja } \mathcal{J} = -\star \mathbf{J}.$$

Olkoon ω on $(m - 1)$ -muoto ja V on m -tilavuus, niin pätee

$$\int_V d \wedge \omega = \int_{\partial V} \omega.$$

Vastaavasti, kun $d\mathbf{V}$ on V :n diff. tilavuusalkio ja $d\mathbf{S}$ pinnan ∂V :n diff. pinta-ala-alkio, pätee

$$\int_V d\mathbf{V} \lrcorner (\partial \wedge \boldsymbol{\omega}) = \int_{\partial V} d\mathbf{S} \lrcorner \boldsymbol{\omega}.$$

kun $\boldsymbol{\omega} \in \bigwedge^{m-1} \mathbb{R}^{p,q}$.

Oppikirjasta ss. 261–264

4§ Variaatiolaskento

Lagrangen formalismi:

$$\mathbf{F} = \partial \wedge \mathbf{A}$$

$$\int_V dV \mathcal{L}$$

$$(dV = dx dy dz c dt)$$

$$\mathcal{L} = \frac{1}{2}(\partial \wedge \mathbf{A}) \lrcorner (\partial \wedge \mathbf{A})$$

Kotitehtävä:

$$\mathcal{C}\ell_{3,1}, SO_+(3,1)$$

$$\langle \mathbf{F} \mathbf{F} \rangle_0$$

$$\mathbf{F} \wedge \mathbf{F}$$

$$\mathbf{F} \wedge \star \mathbf{F}$$

Yhden parametrin avulla määritelty muunnos:

$$\mathbf{x} \rightarrow \mathbf{x}' = \mathbf{x} + \alpha \mathbf{a}$$

$$\mathcal{L}(\mathbf{x}) \rightarrow \mathcal{L}'(\mathbf{x}) = \mathcal{L}(\mathbf{x}')$$

$$\begin{aligned}
\partial_\alpha \mathcal{L}' &= (\mathbf{a} \cdot \partial) \mathcal{L} \\
&= (\partial \wedge (\mathbf{a} \cdot \partial) \mathbf{A}) \lrcorner (\partial \wedge \mathbf{A}) \\
&= \partial \cdot [((\mathbf{a} \cdot \partial) \mathbf{A}) \lrcorner \mathbf{F}] - \underbrace{\dot{\partial} \lrcorner ((\mathbf{a} \cdot \partial) \mathbf{A} \lrcorner \dot{\mathbf{F}})}_{=0, \text{ jos } \dots} \\
&= \partial \cdot [((\mathbf{a} \cdot \partial) \mathbf{A}) \lrcorner \mathbf{F}].
\end{aligned}$$

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Kanoninen energia-liikemääräntensori on esitettävissä vektorina:

$$\mathbf{T}(\mathbf{a}) \equiv [((\mathbf{a} \cdot \partial) \mathbf{A}) \lrcorner \mathbf{F}] - \frac{1}{2} \mathbf{a} \mathcal{L}.$$

Toteuttaa $\partial \cdot \mathbf{T}(\mathbf{a}) = 0$.

Energia-liikemäärä mittamuuttumattomassa muodossa?

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$$\begin{aligned}
\mathbf{T}(\mathbf{a}) &= [\mathbf{a} \lrcorner (\partial \mathbf{A}) - \dot{\partial}(\mathbf{a} \cdot \dot{\mathbf{A}})] \lrcorner \mathbf{F} - \frac{1}{2} \mathbf{a} \mathcal{L} \\
&= (\mathbf{a} \lrcorner \mathbf{F}) \mathbf{F} - [\dot{\partial}(\mathbf{a} \cdot \dot{\mathbf{A}})] \lrcorner \mathbf{F} - \frac{1}{2} \mathbf{a} \mathcal{L} \\
&= (\mathbf{a} \lrcorner \mathbf{F}) \mathbf{F} - \underbrace{\partial \lrcorner ((\mathbf{a} \cdot \mathbf{A}) \mathbf{F})}_{\text{kok.diff.}} - \underbrace{(\mathbf{a} \cdot \mathbf{A}) \partial \lrcorner \mathbf{F}}_{=0, \text{ jos ...}} - \frac{1}{2} \mathbf{a} \mathcal{L}
\end{aligned}$$

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$$\begin{aligned}
\mathbf{T}(\mathbf{a}) &= (\mathbf{a} \lrcorner \mathbf{F}) \mathbf{F} - \frac{1}{2} \mathbf{a} \mathcal{L} \\
&= (\mathbf{a} \lrcorner \mathbf{F}) \mathbf{F} - \frac{1}{2} \mathbf{a} (\mathbf{F} \lrcorner \mathbf{F}) \\
&= -\frac{1}{2} \mathbf{F} \mathbf{a} \mathbf{F} \quad \boxed{\text{Oppikirjan s. 111}}
\end{aligned}$$

Riesz 1947, Hestenes 1966

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Jollekin havaitstijalle $\mathbf{a} = c\mathbf{e}_4$:

$$\mathbf{T}(c\mathbf{e}_4) = -\frac{1}{2}\mathbf{F}c\mathbf{e}_4\mathbf{F} = (\vec{E} \wedge \vec{B})\mathbf{e}_{123} + \frac{1}{2}\left(\frac{1}{c^2}\vec{E}^2 + \vec{B}^2\right)c\mathbf{e}_4$$

Lisälukemista

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Ks. myös

http://www.hut.fi/~ppuska/elmag_alg.html