A practical approach to Monte Carlo estimation of subsurface scattering[1]

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ABSTRACT

In the past over 30 years there has been a tendency in computer graphics to build synthesized pictures that are indistinguishable from real world, seen either by human eye or photographic camera. Until recent days many "ad hoc" or experimental models have been developed that can be used to describe real world illumination and surfaces, for example Phong or Gourad shading. On the other side researchers have observed many materials and surfaces that cannot be modeled by these "ad hoc" methods. Such materials are for example human skin, marble, wet stone etc. As computing power has increased tremendously in recent days there exists a need to develop more accurate models and approach too. Models that are based on classical physical optics.

In this paper we review two models that can be used to analyse subsurface scattering. Typically such surfaces consist of one or several (usually) thin translucent films, coatings or tissues (for example human skin). Each of these layers can be smooth or consists of stochastically distributed micro particles. In either case arriving light will partly scatter and reflect from these layers back to surface and partly penetrate into the material. Finally back scattered and reflected light will accumulate with incoming light. Often this accumulation occurs in different position than original light beam had.

In chapter 1 we qualitatively introduce the topics and problems. Some reflection and scattering properties of materials are discussed in chapter 2. On subsequent chapters we present some fundamental properties of light. Important topics are Fresnel coefficient, Snell's law, lambertian diffusion and electomagnetic nature of light. Based on this fundamental ground we review in chronological order two such model that can be used in subsurface scattering calculations, namely models by Hanrahan and Krueger and most recently presented model by Jensen & al. Finally some conclusions will be drawn. Due to lack of time and space other interesting models will not be presented. For example models by Pharr and Hanrahan and He & al. are not discussed.

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INTRODUCTION

All visible surfaces can be described as sum of emitted and reflected intensity of light to certain direction (directions) [WATT92] Also observed intensity can be described as:

$$I(\theta_r, \phi_r) = I_{emmited}(\theta_r, \phi_r) + I_{reflected}(\theta_r, \phi_r) \tag{1.1}$$

Where
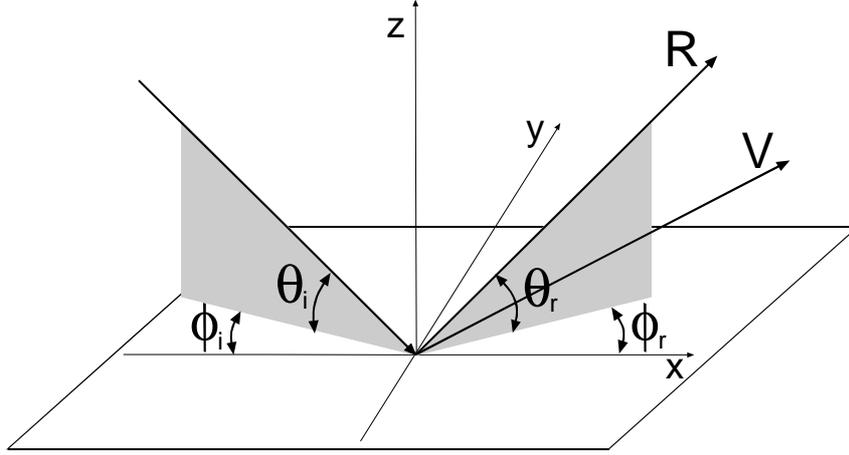θ_r, ϕ_r = outgoing angles from tangent planes

Figure 1: R reflected (mirror) direction V viewing direction

In subsurface scattering theory one of the main idea is that the position of the outgoing light is not necessary the same as coming position. Behind surface of the material can happen diffuse light spreading due to scattering or light can be transported in layered material due to total reflection on the surfaces between the layers etc.

In this paper we consider only reflections thus the emitted term will be neglected. We classify reflections from surfaces as consisting first surface reflections and scattering from subsurface layers or particles. The first surface reflection is strongly directional. On smooth surfaces this reflection is specular or mirror-like and reflection is symmetrical to tangent plane normal. On rough surfaces diffuse like behaviour due to diffraction, scattering and interference effects become more important and overrides the directional behaviour of specular reflection. Diffuse reflection is qualitatively explained as a result of surface or subsurface scattering. Light entering the material is absorbed and scattered and finally exits from the material. Light can be scattered multiple times too. Rough surfaces have been explained consisting of randomly distributed microfacets. As material can be isotropic and roughness can be statistically distributed, the directions in which it leaves the material will be random.

One of the most successful light reflectance model was derived over 25 years ago by Phong. He built an approximate model where the reflectivity is divided to specular and diffuse components (for each light source) [WATT92]

$$\rho_\lambda^{''}(\lambda, \vec{\omega}_r, \vec{\omega}_i) = k_{diffuse} + k_{speculiar} \cos^n \alpha \tag{1.2}$$

where
$\vec{\omega}_{r;i} = [\theta_{r;i}, \phi_{r;i}]^T$

$k_{diffuse}$ = fraction of energy diffusely reflected (in all direction),

$k_{speculiar}$ = fraction of energy specularly reflected,

α = angle between viewing direction (**V**) and true mirror direction (**R**).

The empirical formula of the spread of highlights towards the mirror direction was one of Phong's great innovations. Thus we have cheap and efficient way to calculate geometry of highlights. Phong direct lightning model was a clever scheme using simple and easily calculable representation, but it is neither accurate in true physical sense nor is energy consistent [GREE99, WATT92]. There exist plenty of materials where Phong model simply doesn't work. Human skin, leaves, marble, milk, rough surfaces and wet stone etc. are examples where physical models of reflection instead of empirical "ad hoc" models should be applied

A common approach to modelling surface roughness is to use so-called microfacet model, where surface is still assumed to be clean and consisting of statistically distributed mirror like elements (microfacets). Different statistical methods can then be applied to classical geometric optics in modelling surface roughness. Fresnel equations is used, but weighted by terms, that models surface roughness. For example Davies (already in 1954) uses statistical approach and models surface roughness as RMS mean value when surface heights are Gaussian distributed. Davies defines specular reflectivity as [DAVI54]:

$$D\rho_\lambda{}'(\lambda, \theta_i) \tag{1.3}$$

where

$$D = e^{\left(-\left(4\pi \frac{h}{\lambda} \cos\theta_i\right)^2\right)} \tag{1.4}$$

Blinn use more accurate model by introducing attenuation factor G, due to self shadowing of the microfacets [BLINN77]. Cook further improves Blinn model by paying attention to solid angle of illumination source and incorporating Fresnel colour shift at specular highlights. Cook model has been very successful in modelling metallic objects [GREE99].

None of these models are sufficiently general as they don't take into account polarisation effects, subsurface interaction and light transport in thin layers. In 1991 He & al. introduced sophisticated model based on physical geometric optics. In their model reflections are based on specular, directional diffuse and uniform diffuse reflections [HE91]. He takes into account wavelength, incidence angle, surface roughness parameters and refractive index. Model is very expensive to calculate since it contains slowly convergent infinite summations [GREE99].

1.1 Some definitions

Spectral net irradiance F_ν is defined as net radiative energy flow (power) per unit area within small frequency range $(\nu, \nu + d\nu)$ [STAM01].

$$F_\nu = \frac{d^3 E}{dA\, dt\, d\nu}, \quad [W \cdot m^{-2} \cdot Hz^{-1}]$$

where $d^3 E$ is the flowing net energy. Observe that for electromagnetic radiation wavelength is equal to speed of light divided by frequency of radiation. Integrating over interesting frequency band we get the net *irradiance*

$$F = \int_0^\infty F_\nu d\nu$$

In computer graphics we are often interested in directional dependence of the energy flow. This is calculated by considering small subset of the energy d^4E that flows within solid angle $d\gamma$ around direction $\vec{\omega} = [\theta, \phi]^T$ in the time interval dt and within frequency range $d\nu$. When this energy flows through or reflects from surface element dA we define *spectral radiance*:

$$I_\nu = \frac{d^4E}{\cos(\beta)dAdtd\gamma d\nu}, \quad [W \cdot m^{-2} \cdot sr^{-1} \cdot Hz^{-1}]$$

where β is angle between surface normal and incident beam. Integrating over interesting frequency range (wavelengths) we obtain *radiance*:

$$I = \int_0^\infty I_\nu d\nu$$

Now it is clear that integrating over hemisphere we can calculate irradiance from radiance i.e.:

$$F = \int_{4\pi} \cos(\beta)Id\gamma$$

2.0 MATERIAL PROPERTIES

In this paper all materials are assumed to be composed of one or more layers. These can be semi-infinite in the sense of attenuation of light in the material to certain threshold level (optical depth). Each layer is composed of a mixture of randomly distributed particles embedded in translucent base. These layers can be described by set of parameters [HEN93, STAM01]:

- n index of refraction
- $\sigma_a\,[mm^{-1}]$ absorption cross section
- $\sigma_s\,[mm^{-1}]$ scattering cross section
- $d\,[mm]$ thickness of layer
- $p(\cos(j))$ scattering phase function
- g mean cosine of phase function

n index of refraction
Reflection and refraction are assumed to be results of dielectrictivity of materials (see chapter 3.2)

$\sigma_a\,[mm^{-1}]$ $\sigma_s\,[mm^{-1}]$ *absorption and scattering cross section*
It is quite obvious that the intensity of backscattered and transmitted light depends on the amount of absorption and scattering properties of the material. These properties can be described quantitatively with cross sections measures. Physically these measures can be interpreted as total power per unit area scattered or attenuated divided by power per unit area of the incident beam [STAM01]. Mathematically these cross sections can be interpreted as probability per unit length of interaction [HEN93]. That is the mean free path is equal of inverse of total scattering cross section. Total scattering cross section σ_k is defined as the sum of absorption and scattering cross section. Important concept here is the *albedo* which is defined as scattering cross section divided total scattering cross section:

$$W = \frac{\sigma_s}{\sigma_k} = \frac{\sigma_s}{\sigma_a + \sigma_s}$$

Finally, in some materials these cross-sections have some angular dependencies, thus we can denote $\sigma_k(\alpha), \sigma_s(\alpha), \sigma_a(\alpha)$ where α is the angle between incidence and observation (reflected or transmitted beam) i.e. $\cos(\alpha) = \vec{\omega}_i \bullet \vec{\omega}_{r,t}$.

$p(\cos(j))$ *scattering phase function*

Scattering phase function is defined with the aid of the angular scattering cross section and attenuation factor [STAM01]:

$$p(\cos(\alpha)) = \frac{\sigma(\cos(\alpha))}{\int_{4\pi} d\beta\, \sigma(\cos(\alpha))/4\pi}$$

The scattering phase function represent directional scattering on (subsurface) particle and attenuation in translucent material. If scattering particles are large compared to wavelength of incident radiation, finding interaction properties leads to solving complicated boundary value problems for the electric and magnetic field (i.e. light). p depends on the size, orientation, dielectric properties of particles and the wavelength of incident light. As particle sizes are usually distributed in many different sizes, thus single phase function is not usually very practical. Henley and Greenstein developed empirical formula for phase function to describe diffuse radiation in galaxy. Their formula are applicable here too [HEN93]

$$p(\cos(\alpha)) = \frac{1}{4\pi} \frac{1 - g^2}{\sqrt{[1 + g^2 - 2g\cos(\alpha)]^3}}$$

where
g is the mean cosine of scattered light. When g = 0 material is isotropic and reflection is lambertian (also see chapter 3.1). Positive values indicate forward scattering and negative indicate backward scattering.

Finally if material is composed of several different component then composite material can modelled by simple summation [HAN93]:

$$\sigma_a = \sum_{l=0}^{m} w_l \sigma_{a,l}$$

$$\sigma_s p(\cos(j), g) = \sum_{l=1}^{m} w_l \sigma_{s,l} p(\cos(j), g_l)$$

(1.5)

Incident light will interact with matter: it reflects from and/or refracts into the material. It is found experimentally that the refracting part will attenuate. If the refracting layer has thickness s then the differential loss of radiance is $dI = -kIds$, k is called as *extinction coefficient*. Emphasising wavelength dependence of extinction coefficient we write $k(\lambda)$ and in anistoropic non-homogenous materials we denote $k(x, \lambda)$. Integrating along the beam path we get the reduced *radiance*:

$$I(s, \vec{\omega}) = I(0, \vec{\omega}) e^{-\int_0^s k(x,\lambda)dx}$$

(1.6)

3.0 PHYSICAL PROPERTIES OF LIGHT

We describe here some fundamental properties of light and light-material interaction derived from physical theory of classical ray optics. Our goal is to lay background for further investigations.

According Maxwell's equations electromagnetic waves consists of two perpendicular components. One component is electric field H and the other is magnetic field E. The cross product of these two (Poynting vector) point to the direction of wave propagation. - In practice reflections are calculated using simple "tangent plane approximation". This is done by setting values of the fields - at desired position on the surface - to be the same values that would exit if the surface were replaced by local tangent plane at that point [HE91]. Tangent plane approximation is valid if the curvature of surface is large compared to

the wavelength of the incident electromagnetic wave. In order to keep our calculations and notations as simple as possible, we restricts us in this tangent plane approximation.
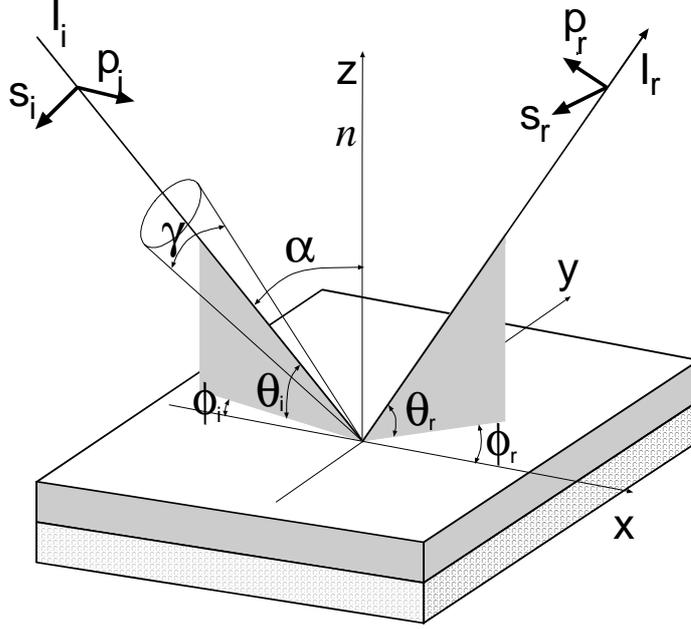

Figure 2: Incoming and outgoing propagation vectors are denoted by I_i and I_r together with polarisation unit vectors s and p (with corresponding subscripts). Surface normal is in z direction and angle between incident light and normal is α Illuminating solid angle is γ

3.1 BRDF and BSSRDF

Here is used such light reflection model that binds incoming energy with outgoing intensity, refereed in the technical literature as bi-directional reflectance distribution function (BRDF). The BRDF, in short the *reflectance*, is defined as the ratio between reflected intensity (cross product of p and s intensities) into the direction of θ_r, ϕ_r to the incident energy per unit time and unit area. The incident energy flux Φ can be expressed in terms of incident intensity and incident solid angle:

$$\rho_r(\vec{\omega}_r, \vec{\omega}_i) = \frac{dI_r(\vec{\omega}_r, \vec{\omega}_i)}{d\Phi_i(\vec{\omega}_i)} = \frac{dI_r(\vec{\omega}_r, \vec{\omega}_i)}{I_i(\vec{\omega}_i)\cos\alpha_i d\gamma_i} = \frac{dI_r(\vec{\omega}_r, \vec{\omega}_i)}{I_i(\vec{\omega}_i)dA_h d\gamma_i}$$

$$\vec{\omega}_i = [\theta_i, \phi_i]^T, \vec{\omega}_r = [\theta_r, \phi_r]^T$$

(1.7)

where $dA_h d\gamma$ relates to the solid angle where energy is contained and to projected differential area. In practice all materials - as well as illuminating light - have some spectral distributions according to light wavelength λ, thus this ratio (1.7) can be written as wavelength depend BRDF formula:

$$\rho_{r\lambda}(\lambda, \vec{\omega}_r, \vec{\omega}_i) = \frac{dI_{\lambda r}(\lambda, \vec{\omega}_r, \vec{\omega}_i)}{\Phi_{\lambda i}(\lambda, \vec{\omega}_i)} = \frac{dI_{\lambda r}(\lambda, \vec{\omega}_r, \vec{\omega}_i)}{I_{\lambda i}(\lambda, \vec{\omega}_i)dA_h d\gamma_h d\lambda}$$

(1.8)

Pure specular reflection occurs when surface is smooth, and diffuse reflection occurs on randomly distributed microscopically irregular surfaces If the reflected radiance from the surface is completely uniform i.e. independent of the angle of observation it is called as Lambert or Lambertian surfaces. From

the bidirectional nature of the light follows that Lambertian surfaces are independent both of the direction of the incidence and observation. Thus reflectance (BRDF) reduces to $\rho_{r\lambda}(\lambda, \vec{\omega}_r, \vec{\omega}_i) = \rho_\lambda(\lambda)$ and we can write Lambert cosine law as [STAM0]:

$$I = \rho_\lambda(\lambda)I_i \cos(\alpha)$$

One main idea in BSSRDF is that position where light enters and leaves is not necessary the same. Thus (1.7) can be rewritten

$$dI_r(x_0, \vec{\omega}_r) = S(x_i, \vec{\omega}_r, x_r, \vec{\omega}_r)dE_i(x_i, \vec{\omega}_i) \tag{1.9}$$

where BSSRDF is denoted with S. Given BSSRDF the output radiance can be simply calculated by integrating over the incoming directions and area A [JEN01]:

$$I_o(x_o, \vec{\omega}) = \int_A \int_{2\pi} S(x_i, \vec{\omega}_i, x_r, \vec{\omega}_r)I_i(x_i, \vec{\omega}_i)(\vec{n} \cdot \vec{\omega})d\omega_i dA(x_i) \tag{1.10}$$

Observe that when incoming and outgoing positions are the same ($x_o = x_I$) then BSSRDF = BRDF [JEN01].

3.2 Reflection from and refraction into dielectric surfaces

When light wave encounters transparent or semitransparent dielectric material it is a known fact that a part of it is reflected and a part of it is refracted (penetrated into the surface). Experimentally it has been found that the fraction of the total energy that is reflected depends on the material constant, index of refraction n_2 of the material [NUSS76].
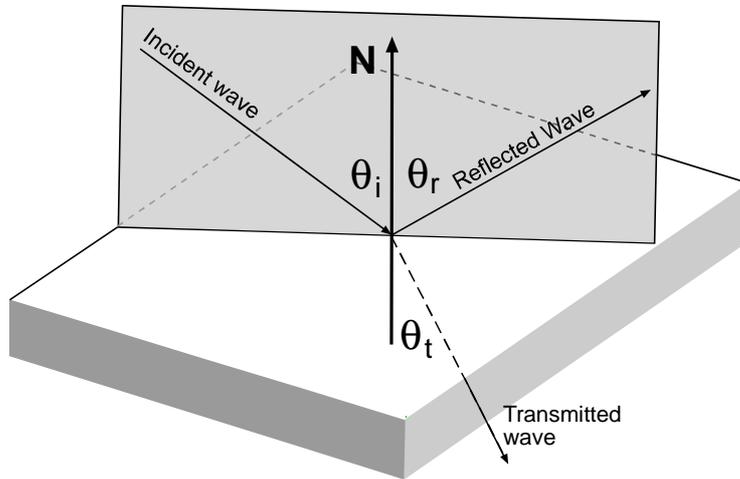

Figure 3 Reflection and refraction on plane surface

In the elementary theory of optics it has been stated that the incident angle and reflected angle are symmetrical in relation to plane normal (N) as a consequence of the tangent plane approximation :

$$\theta_i = \theta_r \tag{1.11}$$

and refraction angle obeys relation (Snell's law)

$$\frac{\sin(\theta_t)}{\sin(\theta_i)} = \frac{n_1}{n_2} \tag{1.12}$$

where n_1 is the refractive index of medium 1 (that contains incident light) and n_2 is the refractive index of medium 2.

Also electromagnetic waves posses polarisation effect. Thus in calculating amount of reflection and refraction we must separate two cases: a) waves that are perpendicular and b) waves that are parallel to incident plane. It can be shown for the reflection and refraction of the *perpendicular polarised light* that for the intensity ratio, for reflection R and refraction T, following equations holds [WATT, WEIS]:

$$R_s = \left[\frac{E_{s,r}}{E_{s,i}}\right]^2 = \left[\frac{\sin(\theta_i - \theta_t)}{\sin(\theta_i + \theta_t)}\right]^2$$

$$T_s = \left[\frac{E_{s,t}}{E_{s,i}}\right]^2 = \left[\frac{2\cos(\theta_i)\sin(\theta_t)}{\sin(\theta_i + \theta_t)}\right]^2 \tag{1.13}$$

and for the *parallel-polarised light* the following ratio holds:

$$R_p = \left[\frac{E_{p,r}}{E_{p,i}}\right]^2 = \left[\frac{\tan(\theta_i - \theta_t)}{\tan(\theta_i + \theta_t)}\right]^2$$

$$T_p = \left[\frac{E_{p,t}}{E_{p,i}}\right]^2 = \left[\frac{2\cos(\theta_i)\sin(\theta_t)}{\sin(\theta_i + \theta_t)\cos((\theta_i - \theta_t)}\right]^2 \tag{1.14}$$

Finally it can be calculated for the non-polarised radiation and for isotropic material Fresnel's coefficients R and T by taking average of perpendicular and parallel reflection and refraction. Also from 1.13 and 1.14 one can easily derive:

$$R = \frac{1}{2}\frac{\sin^2(\theta_i - \theta_t)}{\sin^2(\theta_i + \theta_t)}\left[1 + \frac{\cos^2(\theta_i + \theta_t)}{\cos^2(\theta_i - \theta_t)}\right]$$

$$T = \frac{2\cos^2(\theta_i)\sin^2(\theta_t)}{\sin^2(\theta_i + \theta_t)}\left[1 + \frac{1}{\cos^2(\theta_i - \theta_t)}\right] \tag{1.15}$$

4. HANRAHAN AND KRUEGER'S THEORY

One early attempt to describe the light propagation in thin layers was done by Prahl et al. [PRAH89]. They describe a Monte Carlo algorithm for subsurface scattering of "photons". It is interesting to note that Hanrahan and Krueger's algorithm is similar as Prahl & al's work[2].

As a motivation for this chapter we describe qualitatively the used Monte Carlo algorithm: Light ray (Prahl & al call this misleadingly as "photon") is launched to the surface of the material, part of it is reflected and part of it is refracted according Fresnel's coefficient. Refracted light moves short distance Δs where it may be scattered, absorbed, propagated undisturbed, reflected back to (sub) surface or transmitted out of the layer. The light ray ("photons") is repeatedly moved until all rays ("photons") are

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[2] Hanrahan & Krueger does not even mention Prahl & al even thought they have many factors in common Monte Carlo algorithm for multiple subsequent scattering, Henley-Greenstein phase function to describe material properties etc.. However Hanrahan & Krueger's work goes much further by introducing light transport model and model for thin translucent layers.

either escaped or absorbed. If the light escapes from the surface its position is recorded. If the light is absorbed its position is recorded too. Process is repeated until desired number of rays has been calculated.

Directional and calorimetric properties of diffuse surfaces (Lambertian surfaces) is explained qualitatively as due to subsurface or surface scattering: Light enters the material, partly it is absorbed and partly scattered, depending on the wavelength of the light and colour of the material. In the limit as the light is scattered multiple times it will became isotropic i.e. direction where it will leave the surface is essentially random. This explanation is qualitative and contains no physical parameter. However it contain one fundamental observation: important component of reflection can be composed as a result of subsurface scattering. Hanrahan and Krueger's important contribution to computer graphics was to derive a model for reflection and subsurface scattering using Fresnel's coefficients as a modulating factor.

Hanrahan & Krueger's model is valid on uniformly lit, reasonable flat and homogenous slab. Also it do not take into account local light variations (shades) and curved objects. Besides other things Jensen & al. introduces form factor aimed to extend Hanrahan & Krueger's model [JEN01].

4.1 Reflection from layered surfaces

We will assume that total reflected, observed, radiance I_r from surface consist of two components. One component arises from surface scattering (reflectance) and one from subsurface scattering (see figure 3).

$$I_r(\theta_r, \phi_r) = I_{r,s}(\theta_r, \phi_r) + I_{r,v}(\theta_r, \phi_r) = I_{r,s}(\overline{\omega}_r) + I_{r,v}(\overline{\omega}_r) \qquad (1.16)$$
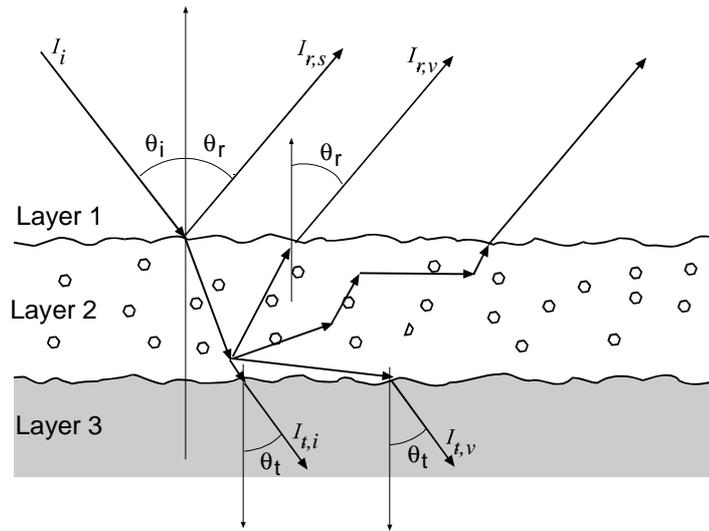

Figure 4: Reflection and scattering in layered material

Similar function as for reflectance (BRDF) can be derived for transmission, named as bi-directional transmission distribution function (BTDF). Thus reflected and transmitted light can be modelled to contain two components [HAN93]:

$$f_r = \rho_{r,s} + \rho_{r,v}$$
$$f_t = \rho_{r,i} + \rho_{t,v} \qquad (1.17)$$

where subscribe r denotes reflected, t transmitted, s surface scattered, v sub surface volume scattered and i incident component.

Finally Fresnell's coefficient modulate both the surface and subsurface terms, thus reflected intensity is:

$$I_r = R\rho_{r,s} + T\rho_{r,v} \tag{1.18}$$

As an immediate consequence from (1.18) is that reflection due to scattering from subsurface is high when Fresnel reflection is low (R is low T is high). Light returning from subsurface layers will partly penetrate through the surface and in some case partly back to subsurface. It will attenuate. In some cases angle of back reflected light can be critical in the sense of Snell's law and thus back reflected light will not be transmitted to the surface boundary, instead it will bounce back

4.1 LIGHT TRANSPORT THEORY

One practical theory to describe propagation of light in matter is light transport theory. It is heuristic and approximate description of electromagnetic scattering and transport. It cannot predict diffraction, interference or different quantum effect. In particular specular reflection of light on rough surface whose height variations are comparable to wavelength of incident light cannot be modelled by this theory [HAN93]. However transport theory has proven its validity on many practical problems for example in modelling organic materials (skin, plant tissue), snow and sand.

Transport theory models the distribution of light in volume with an integro-differential equation of radiance function I:

$$\frac{\partial I(\vec{a},\vec{\omega})}{\partial s} = -\sigma_t I(\vec{a},\vec{\omega}) + \sigma_s \iint p(\vec{a},\vec{\omega},\vec{\omega}')I(\vec{a},\vec{\omega}')d\theta'd\phi' \tag{1.19}$$

where

$\vec{\omega} = (\theta,\phi)$ incident radiance angle on surfaces

$\vec{\omega}' = (\theta',\phi')$ scattered angles

$\sigma_s =$ scattering cross section (see chapter 2)

$\sigma_t =$ transmission cross section (see chapter 2)

p = scattering phase function that represent directional scattering of the light incident onto material (see chapter 2)

$\vec{a} = [x, y, z]^T$ volume (in Cartesian co-ordinates)

This equation (1.19) can be derived by considering energy balance inside differential volume.

If the illumination is relatively constant on the region of interest then interesting component is only z (normal to surface layer). Equation 1.19 can now be simplified by taking into account only z component (neglecting x and y) and we have

$$\cos(\theta)\frac{\partial I(\vec{\omega})}{\partial z} = -\sigma_t I(\vec{\omega}) + \sigma_s \iint p(z,\vec{\omega},\vec{\omega_t})I(\vec{\omega_t})d\theta_t d\phi_t \tag{1.20}$$

Hanrahan and Krueger derived boundary conditions for this differential equation. They concluded that it needs to calculate radiance only on the boundaries z = 0 to the negative direction and on z =d (d = thickness of the layer) to the positive direction. Thus (1.20) reduces further to solving one dimensional transport equation in two directions [HAN93], denoted as I_- (z=0) to negative direction and to positive direction as $I_+(z=d)$.

2.2 SOLVING THE INTEGRAL EQUATION

There exist only few cases where such (1.20) integro-differential equations can be solved directly in closed analytical form. Classic way to solve these are to write them in terms of Neuman series. Physically this can be interpreted as expanding solutions in terms of radiance due to integer number of scattering events. Thus:

$$I = \sum_{k=0}^{\infty} I^{(k)} \tag{1.21}$$

where $I^{(0)}$ is the direct radiance with no (subsurface) scattering, $I^{(1)}$ is scattering due to single scattering event and $I^{(k)}$ is scattering due to k:th scattering event also (1.15) can be written in form

$$\frac{\partial I^{(i+1)}(\vec{a}, \vec{\omega})}{\partial s} = -\sigma_t I^{(i)}(\vec{a}, \vec{\omega}) + \sigma_s \iint p(\vec{a}, \vec{\omega}, \vec{\omega}_t) I^{(i)}(\vec{a}, \vec{\omega}_t) d\theta_t d\phi_t \tag{1.22}$$

and in one dimensional case according Hanrahan and Krueger [HAN93] this can be manipulated to equivalent form :

$$I^{(k+1)}(z, \vec{\omega}) = \int_0^z e^{-\int_0^{z'} \sigma_t \frac{dz''}{\cos(\theta)}} \iint \frac{\sigma_s(z') p(z', \vec{\omega}, \vec{\omega}') I^{(k)}(z', \vec{\omega}')}{\cos(\theta)} d\theta' d\phi' \tag{1.23}$$

2.2.1 First order approximation

In the 0-order case it is assumed that light is scattered or refracted in the medium. Also in the refracted case light is absorbed and attenuated, resulting reduced intensity and we can use directly equation 1.6 here (as expected, see chapter 2 too). For the reason of notational simplicity it is assumed that extinction coefficient is constant. Also for the transmitted light we obtain the following equation when the Fresnel's constants between materials 1-2 and 2-3 are denoted by T^{12} and T^{23} respectively:

$$I_t^{(0)}(\vec{\omega}_t) = T^{12} T^{23} e^{-kd} I_i(\vec{\omega}_i) \tag{1.24}$$

Substituting this into 1-th order solution can the reflected and the back-scattered intensity be calculated. Finally we obtain the following formula for back-scattered radiance:

$$I_r^{(1)}(\vec{\omega}_r) = \frac{\sigma_s}{\sigma_t} T^{12} T^{21} p(\pi - \theta_r, \phi_r, \vec{\omega}_i) \frac{\cos(\theta_i)}{\cos(\theta_i) \cos(\theta_r)} (1 - e^{-\tau_d [1/\cos(\theta_i) + 1/\cos(\theta_r)]}) I_i(\vec{\omega}_i) \tag{1.25}$$

The following conclusion can be done on Hnarahan & Krueger model [HAN93]:
1. Back-scattered light intensity depends on Fresnel's coefficients
2. Reflection steadily increases when layer becomes thicker (transmission to other layers decreases).
3. Subsurface reflection and transmission becomes forward or backward depending on the phase function.
4. Distributions vary as function of reflection direction and it is non Lambertian in nature.
5. Due to Fresnel effect reflection goes to zero at the horizon, as expected.

The above mentioned formulas can be used to generate multiple first order scattering on materials that consist of several layers. On the initial surface the observed total first order scattering is the weighted sum of scattering from each layer. As a weight can be used percentage between the light

refracted to the subsequent layer and returning from that layer. Here must be taken into account the 0-order reduced intensity along the path light travels. Also the Fresnel's coefficient between layers should be take as multiplicative factor in account too.

2.3 THE ALGORITHM

The above process of substituting i-th order solution and then computing (i+1)th solution can be repeated but is very laborious and boring (at least analytically). In subsequent integrals it must be taken in account that angular distributions in scattered radiance comes from the angular properties of the phase function (for each layer). Fortunately light transport in layered media can in practise be calculated using the following Monte Carlo algorithm [HAN93, PRAH89]:

1 Initialize:: Assume that "particle" enters the considered layer at origin. Initialise vectors $\vec{p}$ to the origin and $\vec{s}$ to the direction at which ray enters the layer, set weight w = 1. $\vec{s} = [s_x, s_y, s_z]^T$

2. Events Repeat step **2a** and **2b** until weight of the ray drops below some predefined threshold or ray leaves the illuminated surface:

2a Step:
Estimate the distance to next interaction:

$$d = \frac{\log r}{\sigma_t}$$

where r is uniformly distributed random number [0,1]. Calculate the new position vector:

$$\vec{p} = \vec{p} + d\vec{s}$$

Set particle weight to

$$w = w \frac{\sigma_s}{\sigma_s + \sigma_a}$$

2 b Scatter:

Estimate the cosine of the scattering angle for the Henyey-Greenstein phase function :

$$\cos(j) = \frac{1}{|2g|} \left[1 + g^2 - \left(\frac{1 - g^2}{1 - g + 2gr} \right)^2 \right]$$

calculate $\cos(\phi)$ and $\sin(\phi)$ with $2\pi r$. Calculate the new direction:

$$\vec{t} = \begin{bmatrix} x_s \cos(\phi)\cos(\theta) - y_s \dfrac{\sin(\phi)}{\sin(\theta)} \\ y_s \cos(\phi)\cos(\theta) + x_s \dfrac{\sin(\phi)}{\sin(\theta)} \\ \sin(\theta) \end{bmatrix}$$

$$\vec{s} = \vec{s}\cos(j) + \vec{t}\sin(j)$$

Observe that directions $\cos(\theta) = z_s, \sin(\theta) = \sqrt{1 - z_s}$

3. *Score:* Divide the sphere drawn at incoming position into regions of equal solid angles and add weight of the "particle" to the weight of the weight associated with the bin where it is contained.

4.0 JENSEN & AL'S THEORY

More recently Jensen & al presented at SIGGRAPH 2001 simple model for BSSRDF in translucent materials. Their model enables simulation of traditional scattering and such phenomena as colour bleeding within material and diffusion of light across shadow boundaries. They claim that their model is computationally cheap compared to other existing models.

Main idea on Jensen & al paper is to use diffuse approximation for multiple subsurface scattering. Their approximation is based on the observation that light distribution in highly scattered media (like milk and marble) tends to become isotropic. This is consequence from the fact that each scattering event blurs the light distribution and as result when number of scattering events increases, light distribution tends toward uniformity. In this situation radiance can be approximated with radiant fluence and vector irradiance:

$$I(x,\vec{\omega}) = \frac{1}{4\pi}\varphi(x) + \frac{3}{4\pi}\vec{\omega}\bullet\vec{E}(x) \tag{1.26}$$

Where fluence, or scalar irradiance, is defined as radiant energy per unit area [J/m²] and vector irradiance is defined as radiant power (arriving or leaving) per unit area [W/m²] at specific vector direction, i.e.:

$$\varphi(x) = \int_{4\pi} I(x,\vec{\omega})d\omega$$

$$\vec{E}(x) = \int_{4\pi} I(x,\vec{\omega})\vec{\omega}\,d\omega$$

By considering subsurface scattering (at particles) as a volumetric point source and taking into account Fresnel's reflectance and refractance coefficient we can from diffuse approximation (eq. 1.26) derive equation for diffuse BSSRDF, denoted as R_d. R_d is equal to radiant exitance leaving the surface divided by incident flux (for the algebraic details consult [JEN01]):

$$R_d(r) = -D\frac{(\vec{n}\cdot\vec{\nabla}\varphi)(x_s)}{d\Phi(x_i)} \tag{1.27}$$

where diffusion constant D is defined as $D = \dfrac{1}{3\sigma'_s} = \dfrac{1}{\sigma_s(1-g)}$, also see chapter 2.0 above too.

In general diffusion equation (1.27) does not have explicit, analytic solution. Eason and Farell have developed an approximate method for volumetric source distribution by using dipole sources. In this method two point sources are positioned near the surface: One point source is located at distance z_r beneath the surface and one negative, virtual source, is located above the surface at distance $z_v = z_r + 4\,D$ (see figure 5) [JEN01].
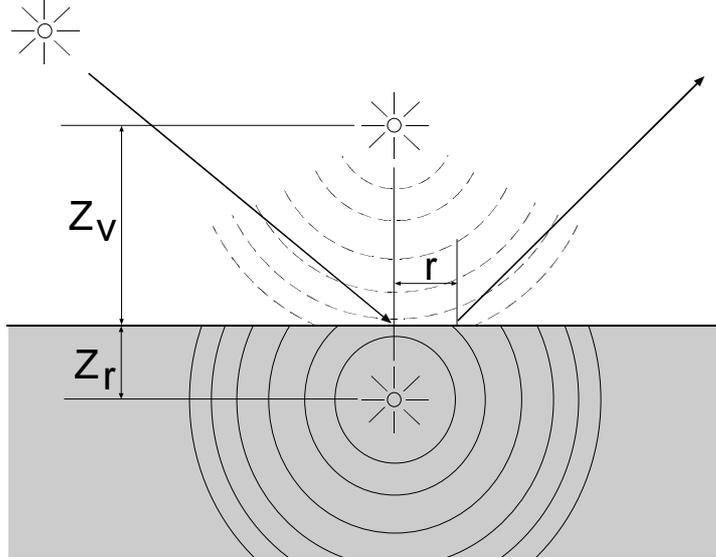
Figure 5: Dipole point source

As fluence will attenuate from these point sources we get the resulting fluence as:

$$\varphi(x) = \frac{\Phi}{4\pi D}\left[\frac{e^{-\sigma_{tr}d_r}}{d_r} - \frac{e^{-\sigma_{tr}d_v}}{d_v}\right] \tag{1.28}$$

where

$d_r = \| x - x_r \|$ is euclidean distance from point x to real source

$d_v = \| x - x_v \|$ is distance from the virtual source

Substituting 1.28 to 1.27 we can calculate diffuse reflectance due to the dipole source

$$R_d(r) = \frac{\sigma'}{4\pi}\left[(\sigma_{tr}d_r + 1)\frac{e^{-\sigma_{tr}d_r}}{\sigma'_t d_r^3} + z_v(\sigma_{tr}d_v + 1)\frac{e^{-\sigma_{tr}d_v}}{\sigma'_t d_v^3}\right] \tag{1.29}$$

Finally, in order to calculate diffusion term S_d in BSSRDF we must take into account the refractive part of incoming light. The Fresnel's coefficient T depict this, thus diffuse BSSRDF can be formulated as:

$$S_d(x_i, \vec{\omega}_i, x_o, \vec{\omega}_o) = \frac{1}{\pi}T(\eta, \vec{\omega}_i)R_d(\| x_i - x_o \|)T(\eta, \vec{\omega}_o) \tag{1.30}$$

4.1 Complete BSSRDF

The complete BSSRDF is the sum of diffuse and single scattering BSSRDF. According Jensen & al the single scattering term can be derived from Hanrahan & Kruegers model by taking into account geometry factor (as explained previously). Also complete BSSRDF can be formulated as:

$$S(x_i, \vec{\omega}_i, x_o, \vec{\omega}_o) = S_d(x_i, \vec{\omega}_i, x_o, \vec{\omega}_o) + S^{(1)}(x_i, \vec{\omega}_i, x_o, \vec{\omega}_o) \tag{1.31}$$

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5.0 CONCLUSIONS

We reviewed here among other things two models on subsurface scattering. In both of these models there has been made strong assumptions compared to classical physical optics:

1. Hanrahan & Krueger approximate optical light theory with light transport theory. It is widely used theory in "computer graphics community" but is heuristic and based on abstracting microscopic phenomena into statistical averages.

2. They model material as turbid layered media with macroscopic scattering and absorption properties. For example modelling biological tissues in "exact" model will involve models for individual cells more detailed (perhaps this is computationally too expensive)

3. Hanrahan & Kruegers model doesn't take into account local illumination variations and assume that the rendered object is reasonable flat.

4. Jensen & al's model don't take into account layered media. They assume that media is semi-infinite and homogenous in relation to attenuated optical length.

5. From the dipole approximation follows that geometry of the media can not be arbitrary. Only locally flat surfaces can be modelled.

6. Both models suffer from spatial texture variations of the object surface. However by modulating cross sections $\sigma_l(x_k), l \in \{s, a, t\}, k \in \{0, i \mid i = 1, 2, 3...\}$ one can in some extent use volumetric textures.

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