Dispersion analysis of a strain-rate dependent ductile-to-brittle transition model

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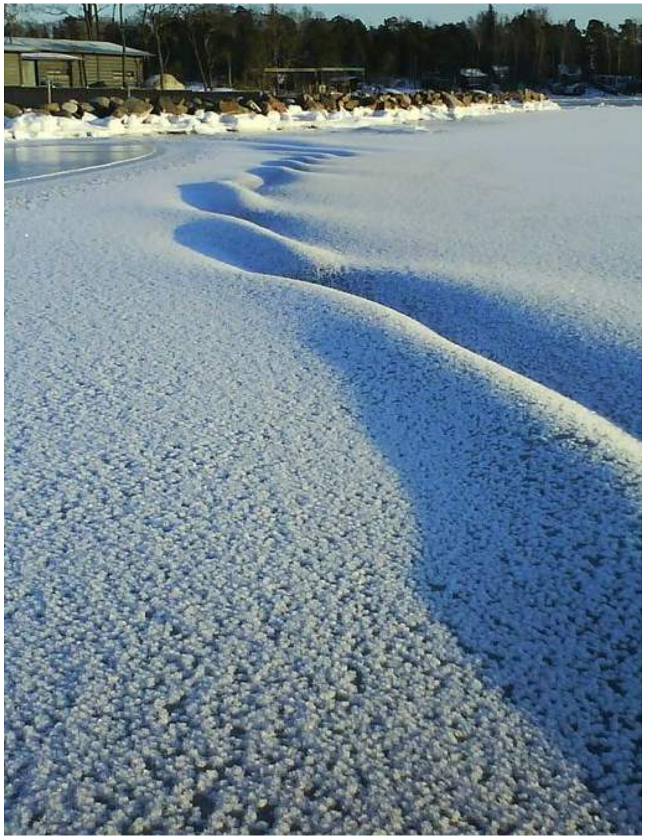

Photograph: Kari Kolari, Helsinki, Feb 2007

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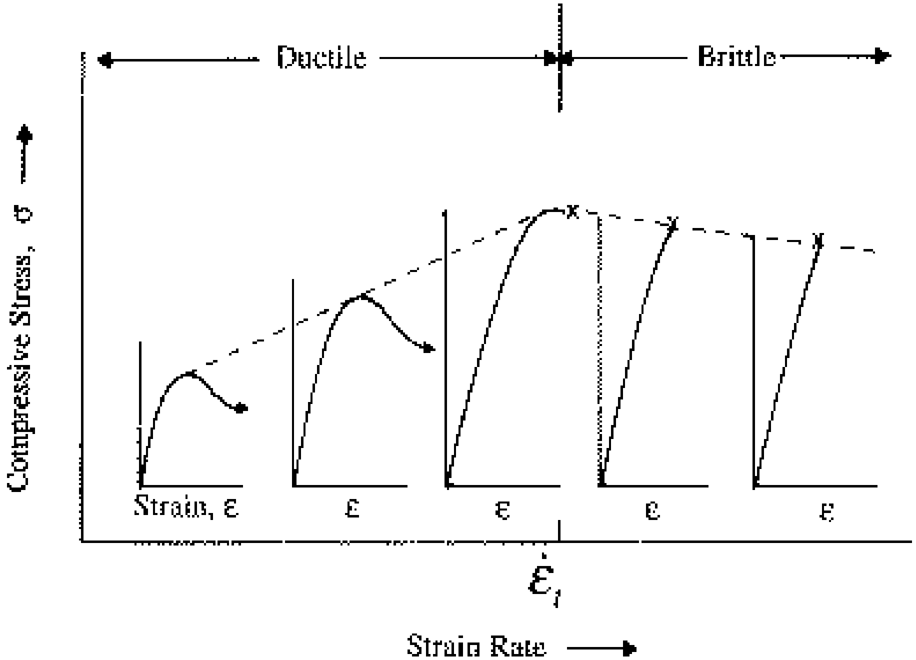
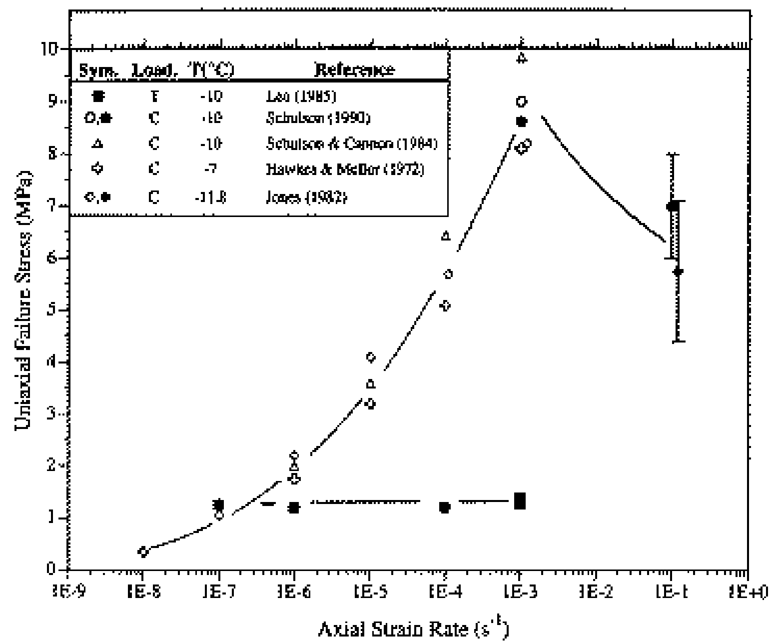
MOTIVATION





E. M. Schulson: Brittle failure of ice, *Engineering Fracture Mechanics* **68** (2001) 1839–1887.

THE MODEL (1-D CASE)

$$\begin{cases} \sigma & = \rho \dfrac{\partial \psi}{\partial \epsilon} = \beta E \epsilon^{\mathrm{e}} = \beta E \left(\epsilon - \epsilon^{\mathrm{i}} \right) \\ \dfrac{\mathrm{d}\epsilon^{\mathrm{i}}}{\mathrm{d}t} & = \dfrac{\partial \varphi}{\partial \sigma} = \left[\dfrac{\varphi_{\mathrm{d}}}{(t_{\mathrm{vp}}^{\mathrm{ps}} \eta)^n \beta \sigma_{\mathrm{r}}} \left(\dfrac{|\sigma|}{\beta \sigma_{\mathrm{r}}} \right)^{np-1} + \dfrac{1}{t_{\mathrm{vp}}^{\mathrm{ps}} \beta} \left(\dfrac{|\sigma|}{\beta \sigma_{\mathrm{r}}} \right)^{p} \right] \operatorname{sign} \left(\dfrac{\mathrm{d}\epsilon}{\mathrm{d}t} \right) \\ \dfrac{\mathrm{d}\beta}{\mathrm{d}t} & = -\dfrac{\partial \varphi}{\partial Y} = -\dfrac{\varphi_{\mathrm{tr}}}{t_{\mathrm{d}} \beta} \left(\dfrac{Y}{Y_{\mathrm{r}}} \right)^{r} \end{cases}$$

$\varphi_{\mathrm{tr}} \geq 0 \quad \varphi_{\mathrm{tr}} \approx 0$ when $\|\dot{\epsilon}^{\mathrm{i}}\| < \eta$ and $\varphi_{\mathrm{tr}} > 1$ when $\|\dot{\epsilon}^{\mathrm{i}}\| > \eta$

$$\varphi_{\mathrm{tr}} = \frac{1}{pn} \left[\frac{1}{t_{\mathrm{vp}}^{\mathrm{ps}} \eta} \left(\frac{|\sigma|}{\beta \sigma_{\mathrm{r}}} \right)^{p} \right]^{n} \qquad \sim \frac{\|\dot{\epsilon}^{\mathrm{i}}\|}{\eta}$$

$$\varphi_{\mathrm{d}} = \frac{1}{r+1} \frac{Y_{\mathrm{r}}}{t_{\mathrm{d}} \beta} \left(\frac{Y}{Y_{\mathrm{r}}} \right)^{r+1}, \quad Y = \frac{1}{2} E (\epsilon^{\mathrm{e}})^2 = \frac{1}{2E} \left(\frac{\sigma}{\beta} \right)^2$$

DISPERSION ANALYSIS - Basics

Dispersion = waves of different wavelengths have different phase speeds

Analysis = put the harmonic wave into the equation of motion

$$u(x,t) = A \exp\left[\mathrm{i}(kx - \omega t)\right],$$

$$\rho \frac{\mathrm{d}^2 u}{\mathrm{d}t^2} - \frac{\mathrm{d}\sigma}{\mathrm{d}x} = 0$$

Dispersion relation $\omega = \Omega(k)$

Phase velocity $v = \dfrac{\omega}{k}$, group velocity $v_{\mathrm{R}} = \dfrac{\partial \omega}{\partial k}$

Nice illustration by Greg Egan

DISPERSION ANALYSIS - Basics

Non-dimensional quantities:

$$\tau = t/t_{\mathrm{e}}, \quad t_{\mathrm{e}} = L/c_{\mathrm{e}}, \quad \text{where} \quad c_{\mathrm{e}} = \sqrt{E/\rho}$$

$$\xi = x/L, \quad \bar{u} = u/L, \quad s = \sigma/\sigma_{\mathrm{r}}$$

Relative strain $e = \epsilon/\epsilon_{\mathrm{r}}$ where $\epsilon_{\mathrm{r}} = \sigma_{\mathrm{r}}/E$

Non-dimensional equation of motion

$$\frac{\mathrm{d}^2\bar{u}}{\mathrm{d}\tau^2} - \epsilon_{\mathrm{r}}\frac{\mathrm{d}s}{\mathrm{d}\xi} = 0, \quad \text{simply} \quad \ddot{u} - \epsilon_{\mathrm{r}} s' = 0$$

Non-dimensional constitutive equations

$$\begin{cases} s & = {\epsilon_{\mathrm{r}}}^{-1}\beta\epsilon^{\mathrm{e}} = {\epsilon_{\mathrm{r}}}^{-1}\beta(\epsilon - \epsilon^{\mathrm{i}}) \\ \dot{\epsilon}^{\mathrm{i}} & = f(\beta, s) \\ \dot{\beta} & = g(\beta, s) \end{cases}$$

DISPERSION ANALYSIS - Viscous material

Constitutive equations: $\dot{s} = {\epsilon_{\mathrm{r}}}^{-1}(\dot{\epsilon} - \dot{\epsilon}^{\mathrm{i}}), \qquad \dot{\epsilon}^{\mathrm{i}} = (\tau_{\mathrm{vp}}^{\mathrm{ps}})^{-1} s^{p}$

Linearization at s_*: $\dot{s}' = \epsilon_{\mathrm{r}}\dot{\epsilon} - a s'$, where $a = \frac{p}{\tau_{\mathrm{vp}}} s_*^{p-1}$

Equation of motion: $\dddot{\bar{u}} - \dot{\bar{u}}'' + a\ddot{\bar{u}} = 0$

Damped harmonic wave: $\bar{u}(\xi, \tau) = \bar{A}\exp(-\bar{\alpha}\xi)\exp[i(\bar{k}_{\mathrm{r}}\xi - \bar{\omega}\tau)]$

Dispersion relation: $i\bar{\omega}(\bar{\omega}^2 - \bar{k}_{\mathrm{r}}^2 + \bar{\alpha}^2) + 2\bar{\omega}\bar{\alpha}\bar{k}_{\mathrm{r}} - a\bar{\omega}^2 = 0$

Solution

$$\bar{\omega} = \frac{\bar{k}_{\mathrm{r}}}{\sqrt{1 + \frac{1}{4}(a/\bar{k}_{\mathrm{r}})^2}}, \quad \bar{\alpha} = \frac{a}{\sqrt{2}\sqrt{1 + \sqrt{1 + (a/\bar{\omega})^2}}}$$

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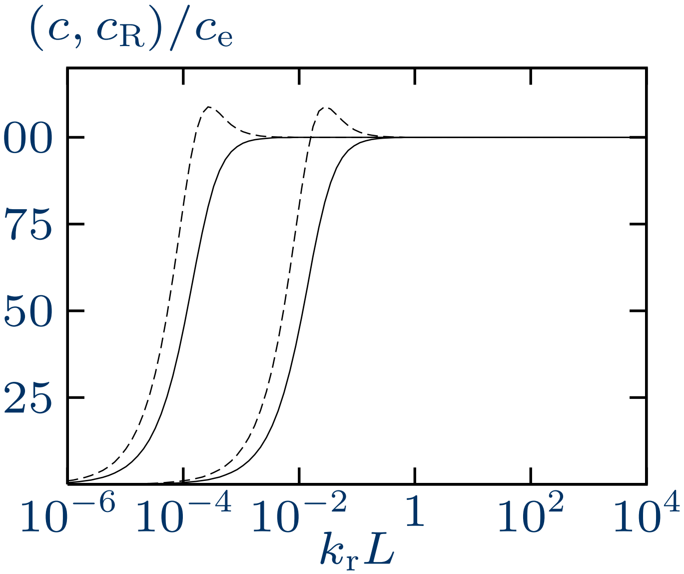
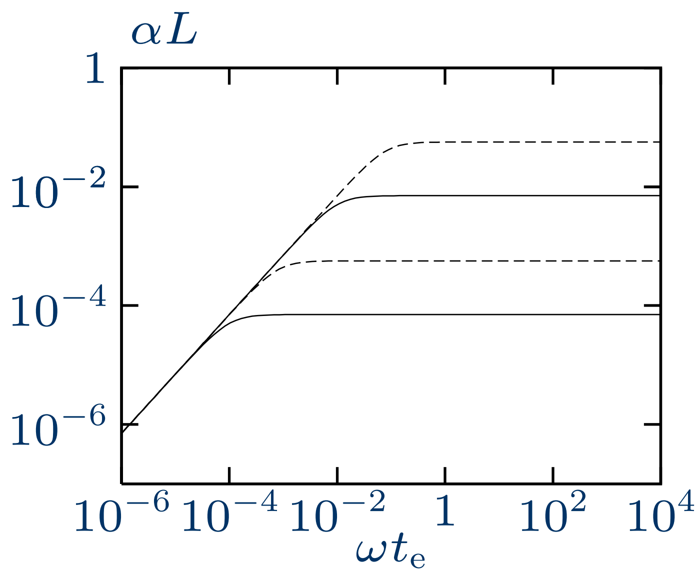
DISPERSION ANALYSIS - Viscous material

$c_{\mathrm{R}} = \dfrac{\mathrm{d}\omega}{\mathrm{d}k_{\mathrm{r}}} = c_{\mathrm{e}}\dfrac{\mathrm{d}\bar{\omega}}{\mathrm{d}\bar{k}_{\mathrm{r}}}$ and $c = \dfrac{\omega}{k_{\mathrm{r}}} = c_{\mathrm{e}}\dfrac{\bar{\omega}}{\bar{k}_{\mathrm{r}}}$ now $c_{\mathrm{R}} > c_{\mathrm{e}}$ anomalous dispersion

$p = 4 \rightarrow \quad \tau_{\mathrm{vp}} = 10^2, 10^4 \leftarrow$ $\qquad p = 1, 8 \uparrow \quad \tau_{\mathrm{vp}} = 10^2, 10^4 \downarrow$





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DISPERSION ANALYSIS - Elastic damaging material

Constitutive equations: $\dot{s} = {\epsilon_{\mathrm{r}}}^{-1}(\dot{\epsilon} - \dot{\epsilon}^{\mathrm{i}}), \qquad \dot{\beta} = -(\tau_{\mathrm{d}})^{-1}\beta^{-2r-1}s^{2r}$

Linearization at s_*, β_* results in the equation of motion:

$$\dddot{\bar{u}} - h_0\dot{\bar{u}}'' - h_1\ddot{\bar{u}} + h_2\bar{u}'' = 0$$

where $h_0 = \beta_*, \quad h_1 = {\tau_{\mathrm{d}}}^{-1}\beta_*^{-2r-2}s_*^{2r}, \quad h_2 = (2r+1){\tau_{\mathrm{d}}}^{-1}\beta_*^{-2r-2}s_*^{2r}$

Dispersion relation:

$$\bar{k}_{\mathrm{r}}{}^4 - a_1\bar{k}_{\mathrm{r}}^2 - a_0^2\bar{\omega}^2 = 0, \quad \bar{\alpha} = a_0\bar{\omega}/\bar{k}_{\mathrm{r}}$$

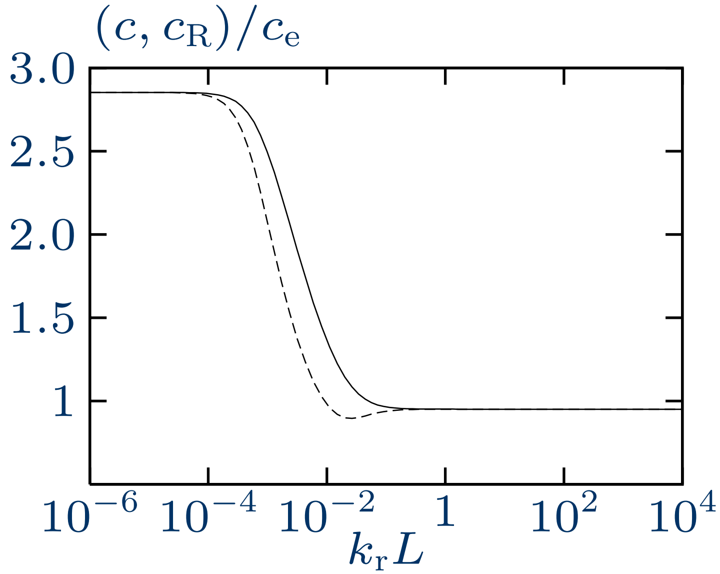
$$a_0 = \frac{(h_2 - h_0h_1)\bar{\omega}^2}{2(h_0^2\bar{\omega}^2 + h_2^2)}, \qquad h_2 - h_0h_1 = \frac{2r}{\tau_{\mathrm{d}}\beta_*}\left(\frac{s_*}{\beta_*}\right)^{2r} > 0$$

$$a_1 = h_0^{-1}(\bar{\omega}^2 - 2h_2a_0)$$

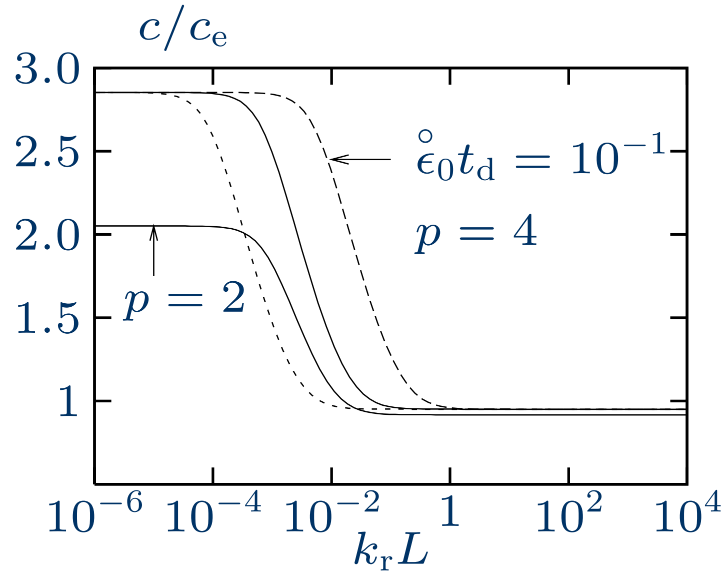
DISPERSION ANALYSIS - Elastic damaging material

$c_\mathrm{R} < c_\mathrm{e}$ normal dispersion

$r = 4, \quad \overset{\circ}{\epsilon}_0 t_\mathrm{d} = 10^{-2}$



$r = 2, 4 \uparrow \quad \overset{\circ}{\epsilon}_0 t_\mathrm{d} = 10^{-1}, 10^{-2}, 10^{-1} \rightarrow$



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DISPERSION ANALYSIS - Full transition model

Equation of motion: $\ddddot{\bar{u}} - h_0 \ddot{\bar{u}}'' - h_1 \dddot{\bar{u}} + h_2 \dot{\bar{u}}'' - h_3 \ddot{\bar{u}} = 0,$

$$h_0 = \beta_* \qquad h_1 = g_\beta + (s_*/\beta_*) g_s - \beta_* f_s$$

$$h_2 = \beta_* g_\beta \qquad h_3 = \beta_* (g_\beta f_s - f_\beta g_s)$$

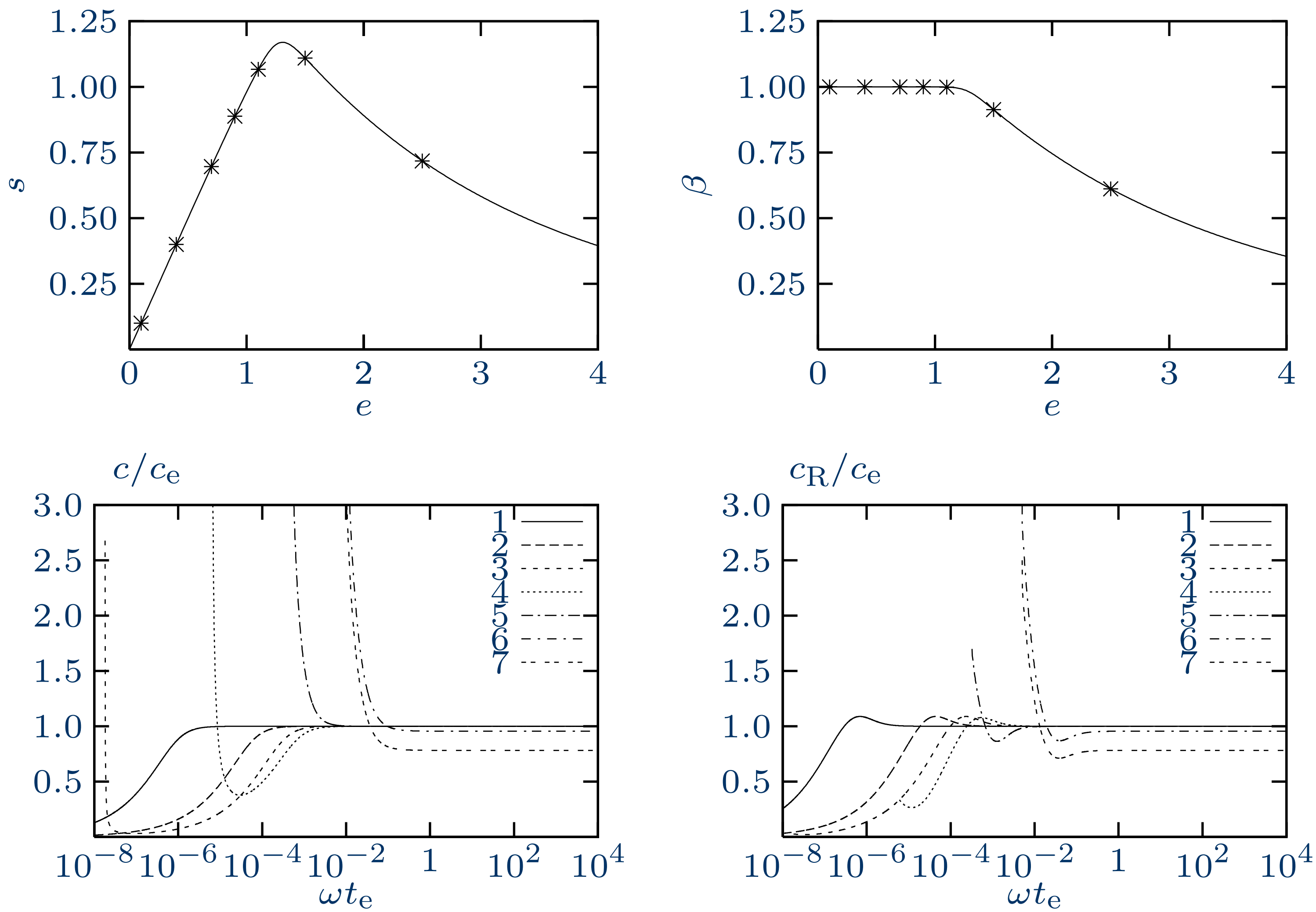
Dispersion relation: $\bar{k}_\mathrm{r}{}^4 - a_1 \bar{k}_\mathrm{r}^2 - a_0^2 \bar{\omega}^2 = 0, \quad \bar{\alpha} = a_0 \bar{\omega} / \bar{k}_\mathrm{r}$

$$a_0 = \frac{(h_2 - h_0 h_1)\bar{\omega}^2 + h_2 h_3}{2(h_0^2 \bar{\omega}^2 + h_2^2)}$$

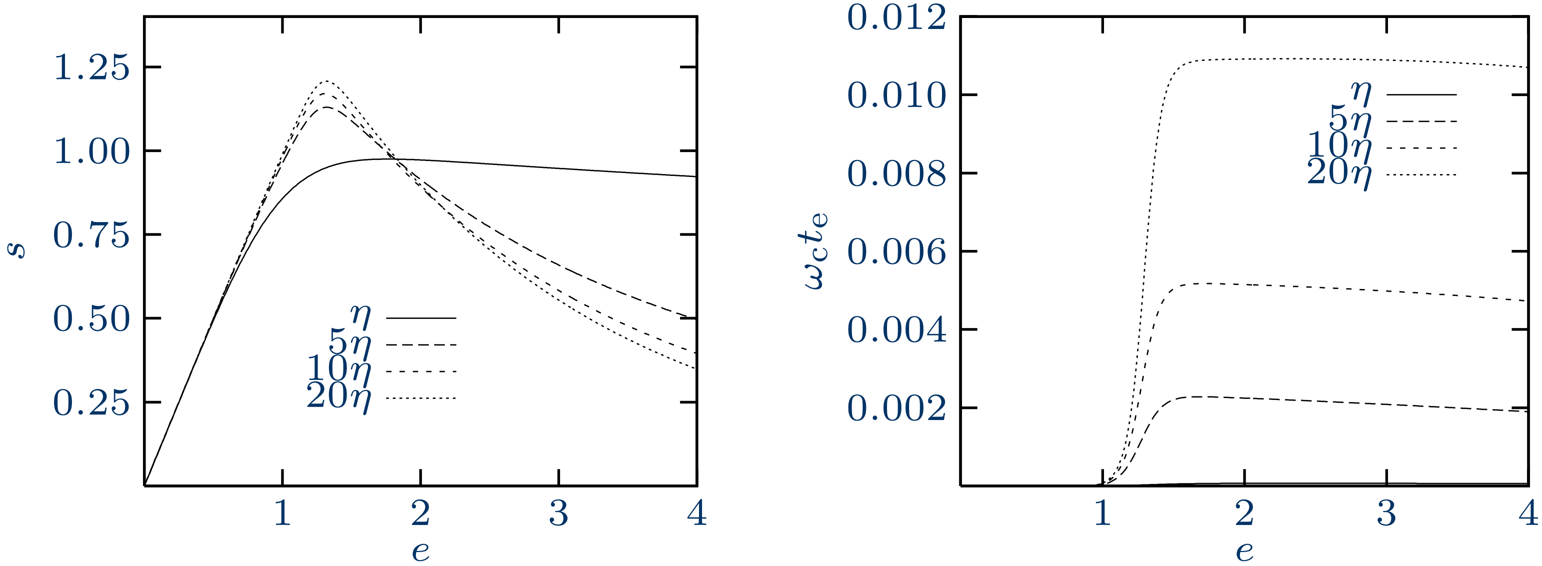
$$a_1 = h_0^{-1} (\bar{\omega}^2 - 2 h_2 a_0 + h_3)$$

DISPERSION ANALYSIS - Full transition model - rate 10η

$E = 40\,\text{GPa}, \sigma_\text{r} = 20\,\text{MPa}, t_\text{vp}^\text{ps} = 1000\,\text{s}, t_\text{d} = 1\,\text{s}, \eta = 10^{-3}\,\text{s}^{-1}, p = r = n = 4$



DISPERSION ANALYSIS - Evolution of the cut-off frequency



Emerges near the peak stress

The saturation value of ω_c depend on the loading rate

CONCLUDING REMARKS

- Both anomalous and normal dispersion depending on state and loading rate
- The model is not able to slow down the high frequency components
- Emerging cut-off frequency
- Length scale of the localization zone?
- Stability?